Higher level Deligne–Lusztig theory: algebraisation, dimension, and sign

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$\mathbb{F}_q$ - a finite field with $\mathrm{char}(\mathbb{F}_q) = p$

G - a connected reductive group over $\mathbb{F}_q$
e.g. $\mathrm{GL}_n, \mathrm{PGL}_n, \mathrm{Sp}_{2n}, \mathrm{SO}_n$ etc.

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L - Lang isogeny $g \mapsto g^{-1}F(g)$

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$B = T \ltimes U$ - Levi decomposition of a Borel subgroup

Assumption: $FT = T$
not necessarily $FU = U$ or $FB = B$

$$G^F = G(\mathbb{F}_q) \curvearrowright L^{-1}(FU) \curvearrowleft T^F$$

translation action

$$\implies \quad G^F \curvearrowright H^i_c(L^{-1}(FU), \overline{\mathbb{Q}}_\ell) \curvearrowleft T^F$$

$H^i_c(-, \overline{\mathbb{Q}}_\ell)$ denotes the compactly supported i-th ℓ-adic cohomology group, $\ell \neq p$ a prime

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Definition

$R^\theta_{T,U} := \sum_i (-1)^i H^i_c(L^{-1}(FU), \overline{\mathbb{Q}}_\ell)_\theta$, where $\theta \in \mathrm{Irr}(T^F)$, is called a Deligne–Lusztig (virtual) representation of G^F.

Theorem (Deligne–Lusztig, 1976)

(1) *If θ is in general position, then $\pm R^{\theta}_{T,U}$ is irreducible;*

(2) *$R^{\theta}_{T,U}$ is independent of the choice of U (hence of B);*

(3) *every irreducible representation of G^F is a subrepresentation of some $\pm R^{\theta}_{T,U}$;*

(4) $\mathrm{Tr}(s, R^{\theta}_{T,U}) = \sum_{w \in W(T)^F} \theta(s^w)$ *for r.s.s. $s \in G^F$;*

(5) $\mathrm{Tr}(1, R^{\theta}_{T,U}) = \dim R^{\theta}_{T,U} = (-1)^{\epsilon_G + \epsilon_T} \cdot \frac{|G^F|_{p'}}{|T^F|}$.

ϵ_H denotes the $\mathbb{F}_q$-rank of an algebraic group H

Example

$G := \mathrm{SL}_2$, so $G^F = \mathrm{SL}_2(\mathbb{F}_q)$.

Let T be such that $T^F \cong C_{q+1}$. Then $L^{-1}(FU) \cong X$, where

$$X\colon \det \begin{bmatrix} x & x^q \\ y & y^q \end{bmatrix} = 1. \quad \text{(an affine plane curve)}$$

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Theorem (Drinfeld, 1974)

Every cuspidal irrep of $\mathrm{SL}_2(\mathbb{F}_q)$ *appears in* $H^1_c(X, \overline{\mathbb{Q}}_\ell)$.

(With multiplicities ≤ 2.)

Our settings:

K - a non-archimedean local field, with residue field $\mathbb{F}_q$
e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p((\pi))$

$\mathcal{O}, \pi$ - its ring of integers, with uniformiser π

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Our focus:

$$\mathrm{SmoothRep}(\mathbb{G}(\mathcal{O})) \quad \leftrightsquigarrow \quad \bigcup_{r \in \mathbb{Z}_{>0}} \mathrm{Rep}(\mathbb{G}(\mathcal{O}/\pi^r))$$

If $r = 1$, we are back to reductive groups over $\mathbb{F}_q$.

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$\exists$ an $\mathbb{F}_q$-algebraic group G_r, with Frobenius F, s.t.

$$G_r(\overline{\mathbb{F}}_q) \cong \mathbf{G}(\mathcal{O}^{\mathrm{ur}}/\pi^r) \quad \text{and} \quad G_r^F \cong \mathbb{G}(\mathcal{O}/\pi^r).$$

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(The so-called Greenberg functor.)

For $\mathbf{B} = \mathbf{TU}$ a Levi decomposition in $\mathbf{G}$, there are also the corresponding algebraic groups B_r, T_r, U_r.

We only concern the case $FT_r = T_r$.

Similar to the $r = 1$ case:

Definition (Lusztig, 1979 Corvallis volume)

$R^{\theta}_{T_r,U_r} := \sum_i (-1)^i H^i_c(L^{-1}(FU_r), \overline{\mathbb{Q}}_\ell)_\theta$, where $\theta \in \mathrm{Irr}(T_r^F)$, is called a higher level Deligne–Lusztig representation of G_r^F.

Recall the $r = 1$ case:

(1) If θ is in general position, then $\pm R^{\theta}_{T,U}$ is irreducible;

(2) $R^{\theta}_{T,U}$ is independent of the choice of U (hence of B);

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Question

The $r > 1$ case?

An overview of the case $r > 1$:

(1)+(2) extends generically: Lusztig(2004), Stasinski(2009).

(3) fails, need to generalise higher DL: Stasinski(2011), C.(2020).
This is the story of my CMS 2019 talk.

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(4 & 5 ++): An explicit formula for $\mathrm{Tr}\left(g, R_{T_r,U_r}^{\theta}\right)$ for $\forall g \in G_r^F$:
C.–Stasinski(2017,2023).
(not available for $r = 1$)

For simplifying the exposition, we assume:

$$\mathbb{G} = \mathrm{GL}_n \quad \text{and} \quad \mathcal{O} = \mathbb{F}_q[[\pi]].$$

All the results, after suitable modifications, work for general $\mathbb{G}(\mathcal{O})$.

From now on let $r > 1$.

Orbits ($r > 1$)

1. Consider the non-degenerate G_r-equivariant bilinear form

$$\mathrm{Tr}\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{A}^1.$$

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2. Fix an (arbitrary) non-trivial character $\mu\colon \mathbb{F}_q \to \mathbb{Q}_\ell^\times$.

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$\Longrightarrow$ (Clifford theory) $\Longrightarrow \exists o'(\rho)$ a G_1^F-orbit in $\mathrm{Irr}(\mathfrak{g}^F)$ s.t.

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via $\mathrm{Tr}(-,-)$ and μ

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Lemma

If $q \to +\infty$, then $\frac{\#\{\text{regular } \theta \in \mathrm{Irr}(T_r^F)\}}{\#\mathrm{Irr}(T_r^F)} \to 1$.

Let $l := [\frac{r+1}{2}]$ and $l' := [\frac{r-1}{2}]$.
So $l + l' = r$, and if r is even then $l = l'$.

Theorem (C.–Stasinski(2017 for even r, 2023 for odd r))

Suppose that θ is regular. Then

$$R^{\theta}_{T_r,U_r} \cong ((-1)^{\epsilon_{G_1}+\epsilon_{T_1}})^r \cdot \mathrm{Ind}^{G_r^F}_{(T_r G_r^{l'})^F} \hat{\rho}_\theta,$$

where $\hat{\rho}_\theta \in \mathrm{Irr}((T_r G_r^{l'})^F)$ is uniquely characterised by:

(a) *If $s \in (T_r^1 G_r^{l'})^F$, then*

$$\mathrm{Tr}(s,\hat{\rho}_\theta) = \left(\frac{1}{q^{n(n-1)/2}}\right)^{l-l'} \cdot \mathrm{Tr}\left(s, \mathrm{Ind}^{(T_r^1 G_r^{l'})^F}_{(T_r^1 G_r^{l})^F} \widetilde{\theta}|_{(T_r^1 G_r^l)^F}\right);$$

(b) *if $s \in T_1^F$, then*

$$\mathrm{Tr}(s,\hat{\rho}_\theta) = \mathrm{St}_{G_1}(s)^{l-l'} \cdot \theta(s).$$

In the above $(-)_r^d := \mathrm{Ker}((-)_r \to (-)_d)$

Remark

If $\mathbb{G}$ is general, not necessarily GL_n:

- replace "regular" by "strongly generic",
- replace "$q^{n(n-1)/2}$" by "$q^{\#\Phi^+}$",
- add "$q \geq 7$".

Remark

This solves a problem raised by Lusztig in 2004 (when $q \geq 7$). Actually, the representation in the RHS was originally constructed by Gérardin; in the proof we found a more conceptual construction.

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Conjecture (C. 2024)

For general $\mathbb{G}$, $\mathbf{T}$, θ, and r:

$$\mathrm{sgn}(R^{\theta}_{T_r,U_r}) = (-1)^{(\epsilon_{G_1}+\epsilon_{T_1})\cdot\left(1+\frac{\log_q |\dim R^{\theta}_{T_r,U_r}|_p}{\#\Phi^+}\right)},$$

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where $(-)_p$ denotes the p-part of a positive integer.

This holds in the following cases:

- $FU_r = U_r$;
- $r = 1$ (Deligne–Lusztig,1976);
- $\mathbb{G} = \mathrm{GL}_n$, and θ is regular (C.–Stasinski,2017,2023);
- $q \geq 7$, and θ is strongly generic (C.–Stasinski,2017,2023);
- $\mathbf{T}$ is elliptic and $\theta = 1$ (C.,2024, and Charlotte Chan 2024);
- $\mathbb{G} = \mathrm{GL}_2$ or SL_2 (C.,2024).

End

Thank you!